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Conservativeness of a Normal Pressure Field Acting on a Shell

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A PROOF has been given by Bolotin[1] that a uniform normal pressure field acting on an arbitrary shell is conservative if, at the edge of the field, the normal displacement or the tangential displacement normal to the edge contour is constrained to zero. This proof consists of deriving the expression for the potential energy Π of the field. The purpose of this note is: 1) to show that the expression for Π given by Bolotin is, in fact, correct only for flat plates; and 2) to generalize the result to a nonuniform continuous normal pressure field acting on an arbitrary shell.

The pressure field shall be defined by the function $p(x^i)$ where x^1, x^2 are curvilinear coordinates of the undeformed shell reference surface S, and x^3 is a normal coordinate measured from S. If, as in Ref. 1, the surface force vector $\tilde{p}_i$, here defined per unit of area of the undeformed surface S, is referred to the undeformed x^i coordinates, then, to the accuracy of linear terms, it follows that

$$\begin{aligned} \tilde{p}_\alpha &= -p\varphi_\alpha \\ \tilde{p}_3 &= -p(1 + e_\alpha{}^\alpha) - u^\alpha p_{;\alpha} - u^3 p_{,3} \end{aligned} \tag{1}$$

where Greek indices take the values 1, 2 and repeated indices are summed; u^i is the displacement vector referred to undeformed x^i coordinates; comma denotes ordinary differentiation with respect to x^i; semi-colon denotes surface covariant differentiation with respect to the metric tensor $a_{\alpha\beta}$ of the undeformed surface; and φ_α and $e_{\alpha\beta}$ are the infinitesimal surface rotation vector and strain tensor, respectively, given by[2]

$$\begin{aligned} \varphi_\alpha &= b_{\alpha\beta}u^\beta - u^3{}_{;\alpha} \\ e_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha;\beta} + u_{\beta;\alpha}) + b_{\alpha\beta}u^3 \end{aligned} \tag{2}$$

where $b_{\alpha\beta}$ is the negative of the second fundamental tensor of the undeformed surface S. From Eqs. (1) and (2), the virtual work of $\tilde{p}_i$ is given by

$$\delta A \equiv \iint_S (\tilde{p}_\alpha \delta u^\alpha + \tilde{p}_3 \delta u^3) dS = -\iint_S \{p(b_{\alpha\beta}\, u^\beta - u^3{}_{;\alpha}) \times \delta u^\alpha + [p(1 + u^\alpha{}_{;\alpha} + b_\alpha{}^\alpha u^3) + u^\alpha p_{;\alpha} + u^3 p_{,3}]\delta u^3\} dS \tag{3}$$

Applying the divergence theorem for a smooth surface, one has

$$\iint_S p u^3{}_{;\alpha}\, \delta u^\alpha dS = -\iint_S u^3 (p\delta u^\alpha)_{;\alpha}\, dS + \int_\Gamma p u^3 \delta u^\alpha n_\alpha d\Gamma \tag{4}$$

where Γ is the edge contour of S, and n_α is a unit outward normal in S to Γ. Substituting Eq. (4) into Eq. (3) and making use of the symmetry of $b_{\alpha\beta}$ gives the result

$$\delta A = -\delta\Pi + \int_\Gamma p u^3 \delta u^\alpha n_\alpha d\Gamma \tag{5}$$

where

$$\Pi = \iint_S \{u^3[p + (pu^\alpha)_{;\alpha}] + \tfrac{1}{2}p b_{\alpha\beta}u^\alpha u^\beta + \tfrac{1}{2}(p b_\alpha{}^\alpha + p_{,3})(u^3)^2\} dS \tag{6}$$

Therefore, the conclusion is that if, at each point of Γ, either p, u^3, or $u^\alpha n_\alpha$ is zero, then the pressure field has the potential energy Π.† Additionally, if, instead of u^3 or $u^\alpha n_\alpha$, any displacement component in their plane is constrained to be zero on Γ, the pressure field still is conservative, but then Π is altered by the addition of a boundary integral term.

In the special case of a uniform pressure field, Eq. (6) reduces to

$$\Pi = p \iint_S [u^3(1 + u^\alpha{}_{;\alpha}) + \tfrac{1}{2}b_{\alpha\beta}u^\alpha u^\beta + \tfrac{1}{2}b_\alpha{}^\alpha (u^3)^2] dS \tag{7}$$

In terms of the notation of this paper, Bolotin's result is

$$\Pi = p \iint_S u^3(1 + u^\alpha{}_{;\alpha} + b_\alpha{}^\alpha u^3) dS \tag{8}$$

which agrees with Eq. (7) only in the case of a flat plate, for which $b_{\alpha\beta} = 0$. The error in Bolotin's derivation arises from a failure to distinguish between surface covariant differentiation and spatial covariant differentiation. In fact, before applying the surface divergence theorem, he failed to convert the expression for the virtual work into terms of surface tensors.

In terms of physical components referred to lines of curvature coordinates, the potential energy is, from Eq. (6),

$$\Pi = \iint_S \left\{p(1 + e_{11} + e_{22})w + \frac{1}{2}\left[p_{,3} - p\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]w^2 + \frac{1}{2}p\left(\frac{u_1{}^2}{R_1} + \frac{u_2{}^2}{R_2}\right) + (u_1 p_{,1} + u_2 p_{,2})w\right\} \alpha_1\alpha_2 dx_1 dx_2 \tag{9}$$

where u_1, u_2, w are displacements; e_{11}, e_{22} are the linearized normal strain expressions; R_1, R_2 are principal radii of curvature; and α_1, α_2 are the Lamé coefficients of the surface coordinates (x_1, x_2).

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† For simplicity in the foregoing derivation for Π, it was assumed that the surface S is smooth. Otherwise, contributions in the form of line integrals along ridges will appear in Π.

Received June 15, 1966.

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Asymptotic Mach Numbers in a Real Gas

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FAR downstream in the inviscid flowfield over long slender blunted bodies, and in the core of wakes behind blunted bodies, the local static pressure is equal to the ambient

Received June 24, 1966.

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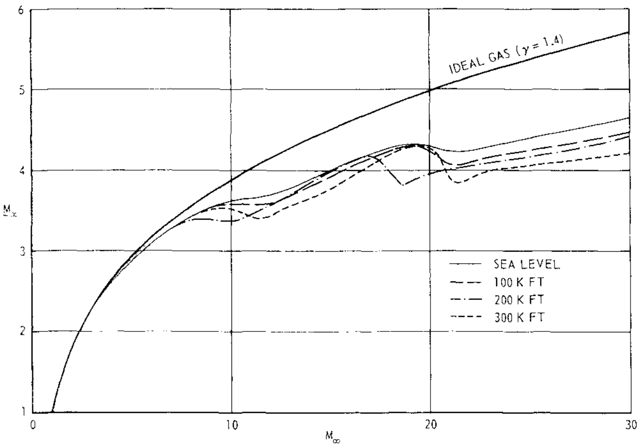

Fig. 1 Asymptotic Mach number for blunt bodies.

atmospheric pressure. Because the flow wetting the body or making up the wake passes through the normal part of the bow shock, a large entropy increase occurs. The static temperature far downstream will then be higher than atmospheric static temperature, and the local velocity will be less than freestream velocity. Thus, the local Mach numbers will be lower than the freestream values.

The asymptotic values of these local "entropy layer" Mach numbers can be determined without knowing the details of the flowfield. Across the shock at the freestream Mach number M_∞, the energy equation holds

$$(u_\infty^2/2) + h_\infty = (u_2^2/2) + h_2 = \text{const} \quad (1)$$

Because of the increase in entropy, h_2 is greater than h_1, and therefore u_2 is less than u_∞, whereas from the shock-wave solution, P_2 is greater than P_∞. By applying the energy equation between the point just behind the shock and far downstream, where $P_{-\infty} = P_\infty$, and making use of the fact that $(S/R)_{-\infty} = (S/R)_2$, the asymptotic velocity $u_{-\infty}$ and asymptotic speed of sound $a_{-\infty}$ are determined. We then have

$$M_{-\infty} = u_{-\infty}/a_{-\infty} \quad (2)$$

This asymptotic Mach number is then useful in checking numerical and theoretical results in regions where the local pressure has nearly reached the ambient pressure.

Figure 1 shows the variation of this asymptotic Mach number with freestream Mach number for several altitudes. The atmosphere used was the 1962 standard,[1] and the local velocity and speed of sound were evaluated by means of a computer program for real-gas normal shocks, coupled to a small program that uses the energy equation to determine the asymptotic velocity. The thermodynamic properties were determined by Hansen's method.[2] The first dip in each real-gas curve in Fig. 1 corresponds to completion of the dissociation of the oxygen molecules; the second corresponds to the completion of the dissociation of the nitrogen molecules. For comparison, the ideal gas curve for $\gamma = 1.4$ is included. Note that the first differences between ideal- and real-gas data appear at Mach numbers between 3 and 4.

The data given in Fig. 1 should provide a rapid check on the operation of both equilibrium and nonequilibrium programs in the far downstream region. Significant deviations from these results are indicative of a possible programing error.

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Comments on "Criteria for Self-Similar Solutions to Radiative Flow"

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In a recent note,[1] Der discussed the similarity criteria for a radiating piston-type problem in the thin and thick limits. This writer wishes to point out that a generalized version of this problem with the thin and thick limits as special cases has been published previously elsewhere.[2] Der's results can be obtained from Eqs. (14a) and (15) of Ref. 2 by setting $\omega = 0$ which corresponds to the case of ambiant density being constant. The symbol n denotes the same in both papers, but β of Ref. (2) corresponds to α in Der's thin criterion and $-\alpha$ in his thick criterion. This is because Der assumed the absorption coefficient as being proportional to $p^\alpha\rho^\beta$ for the thin case and to $p^{-\alpha}\rho^{-\beta}$ for the thick case, whereas this writer considered the same as being proportional to $\rho^\alpha T^\beta$ for arbitrary opacity including the thin and thick limits.

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Comment on "On Dynamic Snap Buckling of Shallow Arches"

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An interesting analysis of the title problem using Galerkin's method has been presented by Humphreys. The mathematical and computational features of the analysis are similar to those which arise in several problems concerning the dynamic behavior of plates and shells. Here we wish to discuss briefly the choice of modal functions used in the Galerkin expansion ϕ_m and the evaluation of the integrals of these functions and their derivatives which arise in the analysis.

In Ref. 1, the infinitely wide (or two-dimensional) arch has been treated using the natural modes of simply supported and clamped beams for the ϕ_m. Humphreys notes that although the integrals of ϕ_m for the simply supported case are handled easily, those for the clamped case are difficult to evaluate numerically because of the problem of small differences between large numbers. It is of interest thus to point out that these integrals have been evaluated analytically by Gallagher and Mercer[2] and also by Ketter.[3,4] Gallagher and Mercer do not indicate their method; Ketter, however, has used a clever variation of integration by parts.

The other point to be made is with regard to the finite width (or three-dimensional) problem. Dowell[5] has carried out such an analysis for the simply supported plate (the extension from the plate to a shallow shell is mathematically trivial though physically important) in a different physical

Received July 5, 1966.

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